ANGLE TRISECTION — COMMENT ON CARL R. WERN'S CONSTRUCTION FOR THE CLASSICAL PROBLEM

The task of trisecting an arbitrary angle using only a (pair of) compass(es) and an unmarked ruler is not entirely unambiguous, since the compass could be demanded to be locked or allowed to be adjustable (e.g. when used for transferring a constructed length). It is true that the technical precision of the drawing deteriorates when adjusting the compass, but even among classical constructions there are those that require compass adjustement as well as those that can be made with a locked compass. The angle bisection belongs to the latter, whereas the circumscribed and inscribed circles of the triangle must be performed with adjustment of the compass. And even if a graduated ruler is not allowed, one can always successively mark equal lengths along a straight line with a locked compass.

The construction of Carl R. Wern implement the trisection at the vertex of an arbitrary angle, although one angle is constructed by transferring a length, using adjustable compass. Cf. Challenge (below). The ruler used is not and will not be marked.

An arbitrary angle V less than about 108° is trisected by Wern's construction into parts equal within one minute of arc, which corresponds to 1 mm along the periphery of a circle with 4 meters radius. (If $108° < V < 180°$, V is at first bisected and what remains to do is obvious. If $180° < V < 360°$, $360° – V$ is bisected at first and what remains to do is as obvious.) For several practical purposes, Wern's trisection could accordingly be considered as perfect. Further, if $V < 12°$, Wern's trisection is accurate to one second of arc.

One will gain clarity by distinguishing direction from angle. Direction is the angle, measured as positive counterclockwise and negative clockwise, from a reference halfline selected at the outset, to the actual halfline. It's number line counterpart is signed distance, from origo to a point. (On the sea, direction is called bearing and is measured as positive clockwise, with north as the reference direction.)

An angle is located between two directions. The value of that angle is the signed difference of two directions. It's number line counterpart is distance. A distance is the difference of two numbers and it can as well be signed. An angle may be oriented arbitrarely and a distance may be located arbitrarely on the number line.

(A sector, finally, is an angle limited by an arc of a circle centered at vertex.)

The angle to be trisected is $\measuredangle \mathsf{ABC}$ (see Fig. 1 in this comment), whose value is denoted by $2v$ below. More specifically, $\measuredangle \mathsf{ABC} = \measuredangle \mathsf{KBC} - \measuredangle \mathsf{KBA} = v - (-v) = 2v$. The trisection is made by constructing two halflines $\mathsf{BR_1}$ and $\mathsf{BW_1}$ such that $\measuredangle \mathsf{KBR_1} \approx v/3$ and

$\measuredangle KBW_1 \approx - v/3$. For sake of clarity, the angle $2v$ in the figure is chosen considerably larger than 108˚, i.e. $2v = 150˚$.

Wern's designations in grotesque (Helvetica) have been retained in this commentary.

F is defined as the intersection between straight lines 5 and 6. E denotes the intersection between circle 1 and circle 7.
The distance AC, normal to the line of symmetry DHBK, intersects this line at a point P. Since the triangles DHF and DPC are congruent, it is realized that

$$\mathrm{HF} = \frac{\sin v}{2 + \cos v} \tag{1}$$

Centered at H, the small circle (7) is drawn through F. A radius in this circle will also form a chord HE in the larger circle 1. HE is then the base of an isosceles triangle ΔHBE with the constructed angle ∡EBH as its vertex angle. Note that ∡EBH = ∡WBK = ∡KBR. The size of each is denoted w. Thus

$$2\sin\frac{w}{2} = \frac{\sin v}{2 + \cos v} \tag{2}$$

whereby in this step of Wern's construction is obtained

$$w = 2\arcsin\left(\frac{1}{2}\frac{\sin v}{2 + \cos v}\right) \tag{3}$$

The difference w–$v/3$ is shown in Fig. 2 and in Table 1. The remarkably high concordance in this step with a perfect trisection up to v≈50˚ is a reason to look for the first terms in a series expansion of the right member in (3)

$$2\arcsin\left(\frac{1}{2}\frac{\sin v}{2 + \cos v}\right) = A_1 v + A_3 v^3 + \ldots \tag{4}$$

(Only odd powers occur.) Note that angles in the following calculations are in radians whilst those in the tables are in degrees.

For expanding in series, first rewrite the right member in (2) as

$$\frac{\sin v}{2 + \cos v} = \frac{\sin v}{3 - (1 - \cos v)} = \frac{1}{3}\frac{\sin v}{1 - \frac{1}{3}(1 - \cos v)} \tag{5}$$

Next, one should expand as a series

$$\frac{1}{1-\frac{1}{3}(1-\cos v)} = 1 + \frac{1}{3}(1-\cos v) + \frac{1}{3^2}(1-\cos v)^2 + \ldots \tag{6}$$

At first, expand

$$1 - \cos v = \frac{v^2}{2} - \ldots \tag{7}$$

$$(1-\cos v)^2 = \left(\frac{v^2}{2} - \ldots\right)^2 = \mathcal{O}\left(v^4\right) \tag{8}$$

Then, (6) may be rewritten as

$$\frac{1}{1-\frac{1}{3}(1-\cos v)} = 1 + \frac{1}{3}\frac{v^2}{2} + \ldots \tag{9}$$

Since

$$\sin v = v - \frac{v^3}{6} + \ldots \tag{10}$$

one will get

$$\frac{\sin v}{1-\frac{1-\cos v}{3}} = \left(v - \frac{v^3}{6} + \ldots\right)\left(1 + \frac{v^2}{3\cdot 2} + \ldots\right) = v + \mathcal{O}\left(v^5\right) \tag{11}$$

The factor 1/3 from (5) is inserted into (11) whereby (2) expands as

$$2\sin\frac{w}{2} = \frac{\sin v}{2+\cos v} = \frac{1}{3}\frac{\sin v}{1-\frac{1-\cos v}{3}} = \frac{v}{3} + \mathcal{O}\left(v^5\right) \tag{12}$$

It should be noted that the third degree term disappears and that this case will not occur if Wern's method would be used for dividing an angle into a number of parts, other than three.
The right member of (12) is inserted into (3)

$$w = 2\arcsin\left[\frac{1}{2}\frac{v}{3} + \mathcal{O}\left(v^5\right)\right] \tag{13}$$

Arcsine expands as $\arcsin t = t + \frac{1}{6}t^3 + \frac{3}{40}t^5 + \frac{5}{112}t^7 + \ldots$ (14)

$$2\arcsin\frac{t}{2} = t + \frac{1}{24}t^3 + \frac{3}{640}t^5 + \frac{5}{7168}t^7 + \ldots \tag{15}$$

Thus

$$w = \frac{v}{3} + \mathcal{O}\left(v^5\right) + \frac{1}{24}\left[\frac{v}{3} + \mathcal{O}\left(v^5\right)\right]^3 + \ldots = \frac{v}{3} + \frac{v^3}{648} + \ldots \tag{16}$$

It can be seen that the third degree coefficient in (16) comes only from the insertion in (15) of the first degree coefficient from (13) and that the third degree coefficient would otherwise not surely have such a small value. This, as well as the fact that only half the vertex angle of the ∡EBH is calculated, can be seen as some kind of reason for the exceptional accuracy of Wern's construction for small angles.

Without derivation, the next two coefficients of the series expansion (4) are stated:

$$A_5 = \frac{-19}{2^7 \cdot 3^4} = \frac{-19}{10368}$$

$$A_7 = \frac{-2753}{2^{10} \cdot 3^7 \cdot 5} = \frac{-2753}{11197440}$$

There are other ways to check the accuracy of the coefficients than actually deriving them, what is left to the reader's ingenuity.

In Fig. 3, which speaks for itself, there are shown graphs

$$2\arcsin\left(\frac{1}{2}\frac{\sin v}{2 + \cos v}\right) - \frac{v}{3} \tag{17}$$

$$A_1 v + A_3 v^3 + A_5 v^5 + A_7 v^7 - \frac{v}{3} = A_3 v^3 + A_5 v^5 + A_7 v^7 \tag{18}$$

$$A_1 v + A_3 v^3 + A_5 v^5 - \frac{v}{3} = A_3 v^3 + A_5 v^5 \tag{19}$$

$$A_1 v + A_3 v^3 - \frac{v}{3} = A_3 v^3 \tag{20}$$

Wern's angle w shall now be used for trisecting the angle $2w$ as shown below. For this purpose, the bisectors BR_1 to ∡WBC and BW_1 to ∡RBA shall be constructed

∡KBC $= v$

∡KBA $= -v$

∡KBR $= w$

∡KBW $= -w$

$$∡KBR_1 = \frac{1}{2}(∡KBW + ∡KBC) = \frac{-w + v}{2} \approx \frac{v}{3} \tag{21}$$

$$\measuredangle \mathrm{KBW}_1 = \frac{1}{2}(\measuredangle \mathrm{KBA}+\measuredangle \mathrm{KBR}) = \frac{-v+w}{2} \approx -\frac{v}{3} \tag{22}$$

$$\therefore \measuredangle \mathrm{KBW}_1 = \frac{1}{2}(\measuredangle \mathrm{KBA}+\measuredangle \mathrm{KBR}) = \frac{-w+v}{2} - \frac{-v+w}{2} = v - w \approx \frac{2v}{3} \tag{23}$$

$$\therefore \measuredangle \mathrm{R}_1\mathrm{BC} = \measuredangle \mathrm{ABW}_1 = \frac{2v-(v-w)}{2} = \frac{v+w}{2} \approx \frac{2v}{3} \tag{24}$$

Note that (21) resp. (22) expresses a direction as the mean value of two directions, (23) expresses an angle as the difference of two directions and (24) expresses an angle as half the difference of two angles.

The trisection deviation after the last step is shown in Fig. 4 and Table 2. [The deviation is halved in (21), (22) and (24).]

Challenge

It is possible to construct Wern's angle w (Eq. 3) without adjusting the compass. The reader may discover for himself how.

Fig. 1. Wern's sketch with the chord HE added.

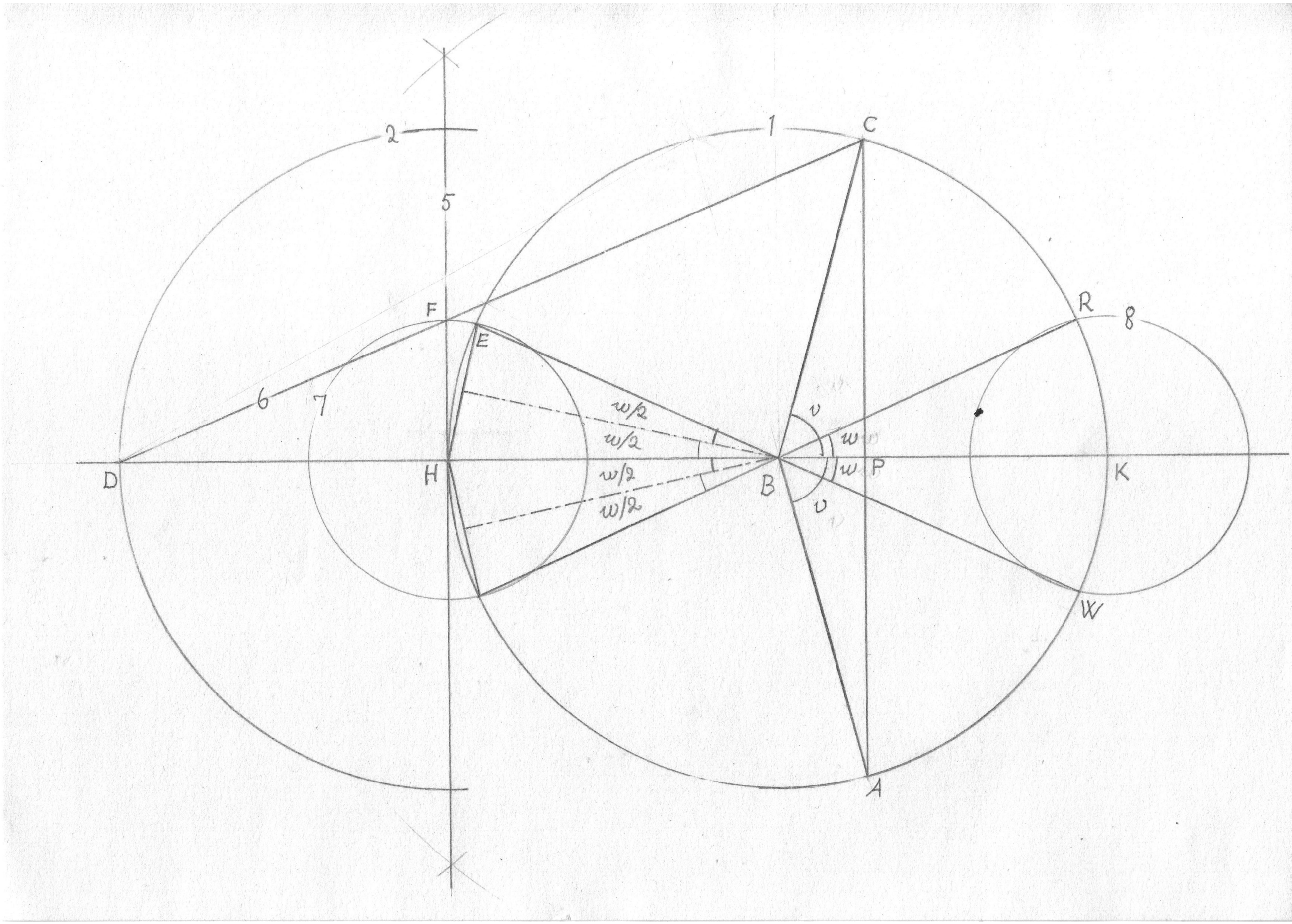

Fig. 2. Deviation from perfect trisection in the first step of Wern's construktion

$w - \frac{v}{3} = 2\arcsin\left(\frac{1}{2}\frac{\sin v}{2 + \cos v}\right) - \frac{v}{3}$ (degrees) as a function of v (degrees)

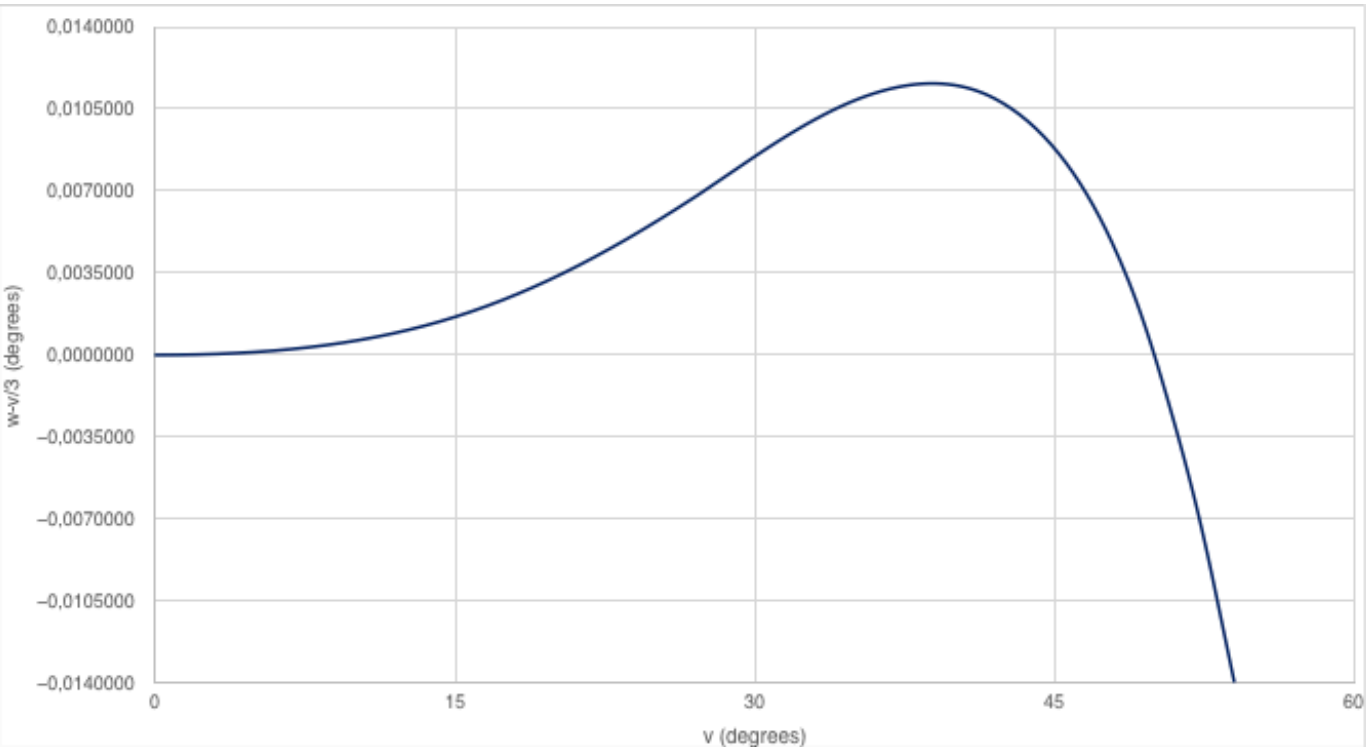

Fig. 3. Deviation in Wern's angle *w* and deviation according to the truncated series (eqs. 17-20). The black curve (eq. 17) is close under the red one (eq. 18).

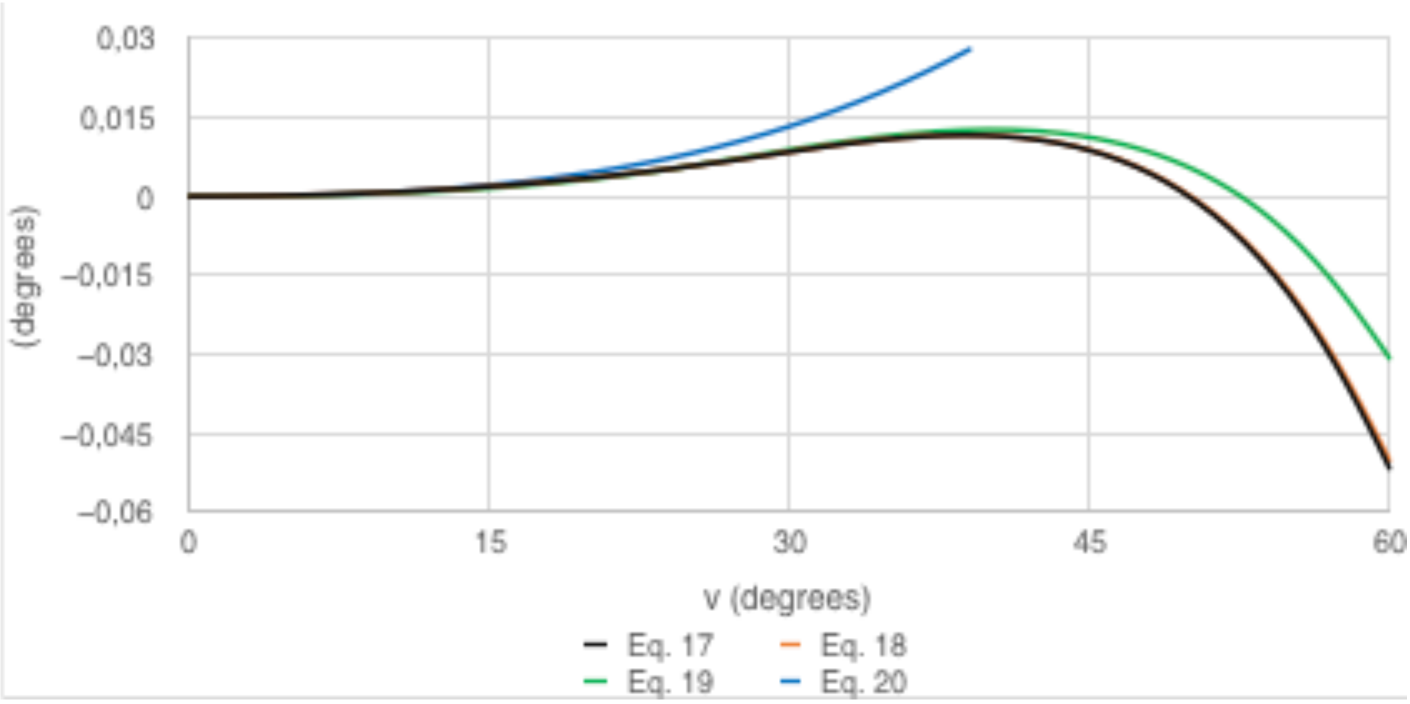

Fig. 4. Deviations from perfect trisection in the second step of Wern's construktion.

$\measuredangle W_1BR_1 - \frac{1}{3}\measuredangle ABC$ (blue)

$\measuredangle R_1BC - \frac{1}{3}\measuredangle ABC$ ($\measuredangle ABW_1 - \frac{1}{3}\measuredangle ABC$) (red)

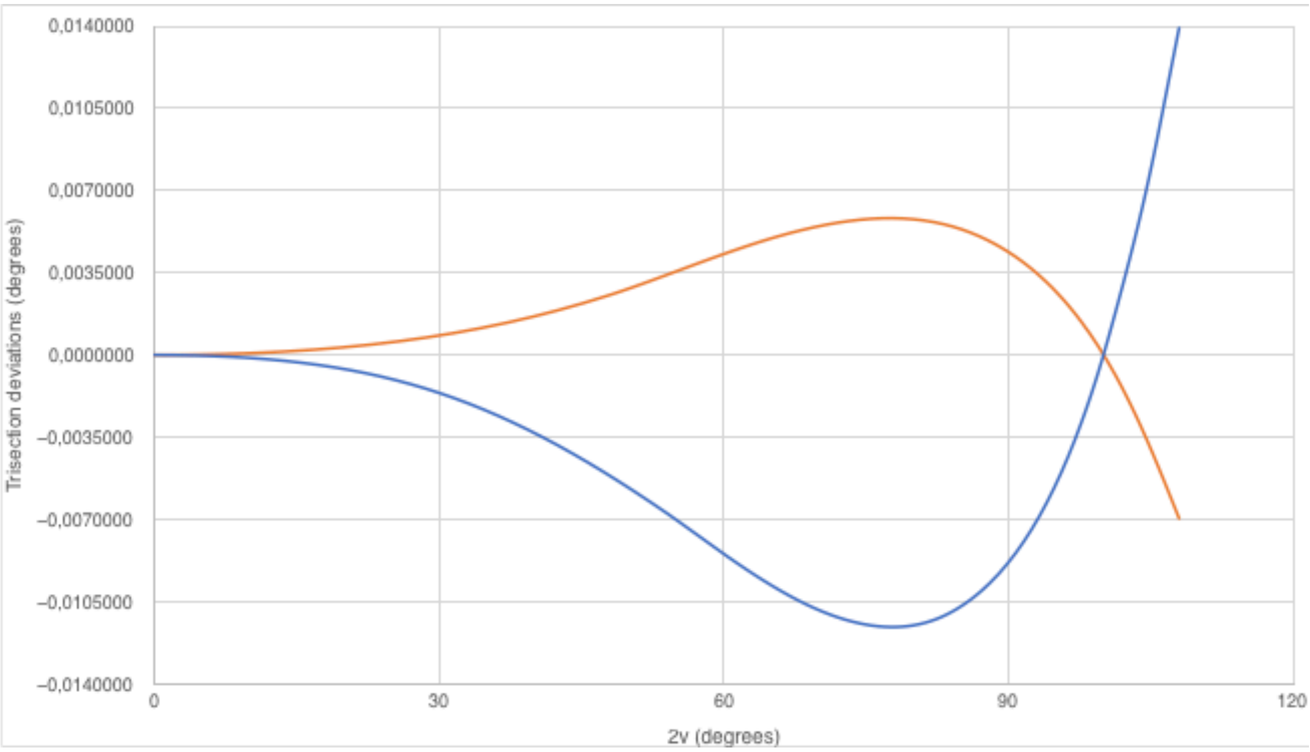

Table 1. Deviation from perfect trisection in the first step of Wern's construktion.

$$w = 2\arcsin\left(\frac{1}{2}\frac{\sin v}{2+\cos v}\right)$$

∡KBC = *v* (deg)	∡KBR = *w* (deg)	*v*/3 (deg)	*w*-*v*/3 (deg)
0	0,000000	0,000000	0,000000
3	1,000013	1,000000	0,000013
6	2,000100	2,000000	0,000100
9	3,000333	3,000000	0,000333
12	4,000770	4,000000	0,000770
15	5,001456	5,000000	0,001456
18	6,002416	6,000000	0,002416
21	7,003646	7,000000	0,003646
24	8,005112	8,000000	0,005112
27	9,006739	9,000000	0,006739
30	10,008405	10,000000	0,008405
33	11,009936	11,000000	0,009936
36	12,011091	12,000000	0,011091
39	13,011560	13,000000	0,011560
42	14,010945	14,000000	0,010945
45	15,008755	15,000000	0,008755
48	16,004389	16,000000	0,004389
51	16,997122	17,000000	–0,002878
54	17,986088	18,000000	–0,013912

Table 2. Deviations from perfect trisection in the second step of Wern's construktion.

$\measuredangle ABC$ (deg)	$\measuredangle W_1BR_1$ (deg)	$\measuredangle R_1BC$ (deg)	$\measuredangle W_1BR_1-(\measuredangle ABC)/3$ (deg)	$\measuredangle R_1BC-(\measuredangle ABC)/3$ $\measuredangle ABW_1-(\measuredangle ABC)/3$ (deg)
0	0,000000	0,000000	0,000000	0,000000
6	1,999987	2,000006	–0,000013	0,000006
12	3,999900	4,000050	–0,000100	0,000050
18	5,999667	6,000166	–0,000333	0,000166
24	7,999230	8,000385	–0,000770	0,000385
30	9,998544	10,000728	–0,001456	0,000728
36	11,997584	12,001208	–0,002416	0,001208
42	13,996354	14,001823	–0,003646	0,001823
48	15,994888	16,002556	–0,005112	0,002556
54	17,993261	18,003369	–0,006739	0,003369
60	19,991595	20,004203	–0,008405	0,004203
66	21,990064	22,004968	–0,009936	0,004968
72	23,988909	24,005546	–0,011091	0,005546
78	25,988440	26,005780	–0,011560	0,005780
84	27,989055	28,005472	–0,010945	0,005472
90	29,991245	30,004377	–0,008755	0,004377
96	31,995611	32,002194	–0,004389	0,002194
102	34,002878	33,998561	0,002878	–0,001439
108	36,013912	35,993044	0,013912	–0,006956

Enskede, Sweden, Aug. 22th, 2022
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